

IV. DETERMINANTS

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Definition

If A is an $n \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix},$$

then the determinant of A , denoted $\det A$, is defined by

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn} \pi \cdot a_{1\pi(1)} \cdot a_{2\pi(2)} \cdot \dots \cdot a_{n\pi(n)}.$$

Notes for better understanding

1. Here π is the symbol for a permutation of the indices of matrix columns. A permutation of $(1, 2, \dots, n)$ is an n -tuple (m_1, m_2, \dots, m_n) that contains each of the numbers $1, \dots, n$ exactly once. The set of all permutations of $(1, 2, \dots, n)$ is denoted S_n . For example, if $S = \{1, 2, 3\}$, then the set S_3 consists of six permutations ($S_3 = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$). We know that S_n has $n!$ elements ($n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$).
2. The symbol $\operatorname{sgn} \pi$ is called the *sign of permutation* (m_1, m_2, \dots, m_n) . The sign of permutation π is defined to be 1 if there is an even number of pairs of integers (j, k) with $1 \leq j < k \leq n$ such that $m_j > m_k$ and -1 if there is an odd number of such pairs. In other words, the sign of a permutation equals -1 if the natural order has been reversed odd number times. For example the permutations $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$ have their signs $1, -1, -1, 1, -1$ and 1 .
3. We will denote the determinant of the matrix A by the symbol

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}, \quad \text{or } \det A, \quad \text{or } |A|.$$

4. In some texts, the determinant is defined as a function on the square matrices that is linear as a function of each row and that it changes the sign when two rows are interchanged, so that the determinant is a multilinear and alternating function on the square matrices. To prove that such a function exists and that it is unique is a non-trivial task.
5. Sometimes we will speak about the „rows or columns of a determinant“ instead of a more precise expression „rows or columns of a matrix from which we calculate the determinant“.

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How to calculate determinant of square matrices of „small“ row ranks

1. If A is 1×1 matrix, $A = (a_{11})$, then $\det A = a_{11}$.

2. If A is 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

Example

We have

$$\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} = 2 \cdot 5 - (-1) \cdot 3 = 10 + 3 = 13.$$

3. If A is a 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

then

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

The above described „algorithm“ is called *Sarrus' rule*.

Example

We have

$$\begin{aligned} \begin{vmatrix} 2 & 3 & 0 \\ -1 & 5 & 1 \\ 0 & 2 & 1 \end{vmatrix} &= 2 \cdot 5 \cdot 1 + 3 \cdot 1 \cdot 0 + (-1) \cdot 2 \cdot 0 - 0 \cdot 5 \cdot 0 - 2 \cdot 2 \cdot 1 - (-1) \cdot 3 \cdot 1 = \\ &= 10 + 0 + 0 - 0 - 4 + 3 = 9. \end{aligned}$$

For practical calculations, the definition of the determinants is not, in general, suitable because $n!$ grows large very rapidly as n increases. For example,

$$\begin{array}{rclcl} n = 2 & S_2 & = & 2! & = & 2 \\ n = 3 & S_3 & = & 3! & = & 6 \\ n = 4 & S_4 & = & 4! & = & 24 \\ n = 5 & S_5 & = & 5! & = & 120 \end{array}$$

We wish to find a more effective method for our computation.

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General theorems

1. If A is a square matrix with one zero row, then $\det A = 0$.
2. If A is a square matrix with two equal rows, then $\det A = 0$.
3. Suppose that A is a square matrix. If B is the matrix obtained from A by interchanging two rows, then $\det B = -\det A$.
4. For every a square matrix A , $\det A = \det A^t$.
5. Suppose that A is a square matrix. If B is the matrix obtained from A by multiplying a row by a scalar λ , then $\det B = \lambda \cdot \det A$.
6. Suppose that A is a square matrix. If B is the matrix obtained from A by multiplying it by a scalar λ , then $\det B = \lambda^n \cdot \det A$.
7. Suppose that A is a square matrix. If B is the matrix obtained from A by adding, say, λ -times the i -th row to the j -th row then, $\det B = \det A$.
8. The determinant of every upper-triangular (or lower-triangular) matrix is equal to the product of all diagonal entries, thus $\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdot \dots \cdot a_{nn}$.
9. The determinant of every diagonal matrix is equal to the product of the diagonal entries, so that $\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdot \dots \cdot a_{nn}$.

Try to prove these theorems using only the definition of the determinant of a matrix.

Theorem

If A and B are square matrices of the same size, then

$$\det AB = \det A \cdot \det B.$$

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The following result shows how we can calculate determinant of $n \times n$ matrix from determinants of $(n-1) \times (n-1)$ matrices.

Definition

Let A be a matrix of size $n \times n$. By the symbol A_{ij} we mean the matrix of size $(n-1) \times (n-1)$ obtained from the matrix A by omitting its i -th row and j -th column. The determinant of matrix A_{ij} is called the *subdeterminant* of matrix A which corresponds to the entry a_{ij} .

Theorem – Laplace expansion along the i -th row (along the j -th column)

Let A be a matrix of size $n \times n$. Then

$$\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{ij},$$

which is called the *Laplace expansion along the i -th row*. Alternatively

$$\det A = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det A_{ij},$$

which is called the *Laplace expansion along the j -th column*.

Note the independence on the row (column) chosen. Note also that on the left hand side, there is just determinant of matrix A of size $n \times n$, on the right hand side there is n determinants of matrices of size $(n-1) \times (n-1)$ which come from the matrix A .

Example

Calculate the determinant of a matrix A , where

$$A = \begin{vmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{vmatrix}.$$

The matrix A is of size 4×4 , so that for its calculation it is impossible to use Sarrus' algorithm. We use preferably Laplace expansion along the 1-th column because it contains many zero entries.

$$\begin{aligned} & \begin{vmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{vmatrix} = \\ & = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} + (-1)^{2+1} \cdot 0 \cdot \begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} + (-1)^{3+1} \cdot 0 \cdot \begin{vmatrix} 1 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} + (-1)^{4+1} \cdot 0 \cdot \begin{vmatrix} 1 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} = \\ & = 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} + 0 + 0 + 0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 12 + 0 + 0 - 4 - 0 - 3 = 5. \end{aligned}$$

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Applications of determinants

In what follows, we will offer the most important applications of determinants.

• The algorithm for calculation of an inverse matrix

Definition

Let A be a matrix of size $n \times n$. We define the *adjugate* matrix of A to be the $n \times n$ matrix $\text{adj } A$ given by

$$(\text{adj } A)_{ij} = (-1)^{i+j} \det A_{ji}.$$

It is very important to note the interchange of the indices in the above definition. The adjugate matrix has the following useful property.

Theorem

Let A be matrix of size $n \times n$, $\text{adj } A$ its adjugate matrix, then

$$A \cdot \text{adj } A = (\det A) \cdot E_{n \times n}.$$

The matrix $(\det A) \cdot E_{n \times n}$ is a diagonal matrix in which every entry on the diagonal is the scalar $(\det A)$. We can use the above result to obtain a convenient way of determining whether or not a given matrix is invertible, and a new way of computing inverses.

Theorem

A square matrix A is invertible, if and only if $\det A \neq 0$, in which case the inverse is given by

$$A^{-1} = \frac{1}{\det A} \cdot \text{adj } A.$$

Note

The above result provides a new way how to compute inverse matrices.

In particular, one should note the factor $(-1)^{i+j} = \pm 1$. The sign is given according to the scheme

$$\begin{array}{ccccccc} + & - & + & - & \dots & & \\ - & + & - & + & \dots & & \\ + & - & + & - & \dots & & \\ - & + & - & + & \dots & & \\ \dots & \dots & \dots & \dots & \dots & & \end{array}$$

Example

Let be A

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 7 \\ 3 & 2 & 1 \end{pmatrix}.$$

Calculate A^{-1} (if exists).

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 5 & 7 \\ 3 & 2 & 1 \end{vmatrix} = 5 + 0 - 21 - 0 - 0 - 14 = -30 \neq 0.$$

Now we know that the matrix A^{-1} exists. We will calculate the *adjugate* matrix of A .

$$\det A_{11} = \begin{vmatrix} 5 & 7 \\ 2 & 1 \end{vmatrix} = -9 \quad \det A_{21} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1 \quad \det A_{31} = \begin{vmatrix} -1 & 0 \\ 5 & 7 \end{vmatrix} = -7$$

$$\det A_{12} = \begin{vmatrix} 0 & 7 \\ 3 & 1 \end{vmatrix} = -21 \quad \det A_{22} = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1 \quad \det A_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 7 \end{vmatrix} = 7$$

$$\det A_{13} = \begin{vmatrix} 0 & 5 \\ 3 & 2 \end{vmatrix} = -15 \quad \det A_{23} = \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 5 \quad \det A_{33} = \begin{vmatrix} 1 & -1 \\ 0 & 5 \end{vmatrix} = 5$$

So $\text{adj } A$ is

$$\text{adj } A = \begin{pmatrix} -9 & 1 & -7 \\ 21 & 1 & -7 \\ -15 & -5 & 5 \end{pmatrix}.$$

The inverse matrix of A is

$$A^{-1} = -\frac{1}{30} \cdot \begin{pmatrix} -9 & 1 & -7 \\ 21 & 1 & -7 \\ -15 & -5 & 5 \end{pmatrix}.$$

In order to calculate the inverse matrix of matrix A of size 3 we had to compute one determinant of a matrix A and 9 determinants of adjugate matrices of matrix A .

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- **The algorithm for solving system of linear equations with a regular matrix**

Theorem (so called Cramer's rule)

Let be $Ax^t = b^t$ non-homogeneous system of linear equations where A is a matrix $n \times n$ of rank n (so that A is a regular matrix), $b \neq o$. Then the system has only one solution (x_1, x_2, \dots, x_n) given by

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, \dots, n,$$

where A_i is a matrix which formed from the matrix A by exchanging the i -th column of matrix A by the column b^t .

Example

Solve the following system of linear equations

$$\begin{aligned}x + y + z &= 1, \\x - y &= 2, \\x - z &= 0.\end{aligned}$$

We will use Cramer's rule.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3,$$

$$\det A_x = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 3, \quad \det A_y = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -3, \quad \det A_z = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 3.$$

$$x = \frac{\det A_x}{\det A} = \frac{3}{3} = 1, \quad y = \frac{\det A_y}{\det A} = \frac{-3}{3} = -1, \quad z = \frac{\det A_z}{\det A} = \frac{3}{3} = 1.$$

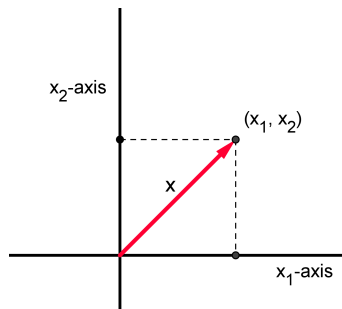
So the solution of our system of linear equations is $[1, -1, 1]$.

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Some geometrical applications of vector spaces, matrices and determinants**• Dot product**

To motivate the concept of inner product, let us consider vectors in \mathbb{R}^2 and \mathbb{R}^3 as arrows with initial point at the origin. The length of a vector $x \in \mathbb{R}^2$ or \mathbb{R}^3 is called the *norm* of x , denoted $\|x\|$. Thus for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$



Similarly, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The generalization to \mathbb{R}^n is obvious.

Definition

We define the *norm* of a vector x , $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

Definition

For all $x, y \in \mathbb{R}^n$ we define the *dot product* of x and y , denoted $x \cdot y$, by

$$x \cdot y = x_1y_1 + \dots + x_ny_n,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

Notes

1. Note that the *dot product* of two vectors in \mathbb{R}^n is a number, not a vector.
2. Obviously $x \cdot x = \|x\|^2$ for all $x \in \mathbb{R}^n$.
3. In particular, $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$, with equality if and only if $x = 0$.
4. Also, if $y \in \mathbb{R}^n$ is fixed, then clearly the map from \mathbb{R}^n to \mathbb{R} sending $x \in \mathbb{R}^n$ to $x \cdot y$ is linear.
5. Furthermore, $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}^n$.

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• Inner product

The notion of inner product is a generalization of the dot product.

Definition

An *inner product* on a vector space V is a function that maps each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in F$ and has the following properties:

1. *positivity*: $\langle v, v \rangle \geq 0$ for all $v \in V$;
2. *definiteness*: $\langle v, v \rangle = 0$ if and only if $v = 0$;
3. *additivity in the first variable*: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$;
4. *homogeneity in the first variable*: $\langle av, w \rangle = a \langle v, w \rangle$ for all $a \in F$ and all $v, w \in V$;
5. *conjugate transpose*: $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$.

Recall that every real number equals its complex conjugate. Thus if we are dealing with a real vector space, then in the last condition we can simply state that $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$.

Note

Note in for \mathbb{R}^n , the dot product is an inner product.

Definition

An *inner product space* is a vector space V equipped with an inner product on V .

Definition

Two vectors $u, v \in V$ are said to be *orthogonal*, if $\langle u, v \rangle = 0$.

Note that the order of the vectors does not matter because $\langle u, v \rangle = 0$, if and only if $\langle v, u \rangle = 0$. Instead of saying that u and v are orthogonal, sometimes we say that u is orthogonal to v . Clearly o is orthogonal to every vector. Furthermore, o is the only vector that is orthogonal to itself.

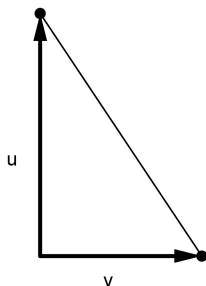
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- **Some known theorems**

Pythagorean theorem

If u, v are orthogonal vectors in V , then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

**Cauchy-Schwarz inequality**

If u, v are vectors in V , then

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|.$$

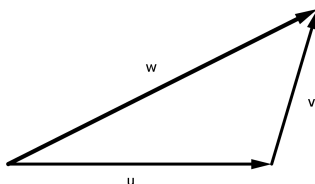
The equality holds if and only if one of u, v is a scalar multiple of the other.

Triangle inequality

If u, v are vectors in V , then

$$\|u + v\| \leq \|u\| + \|v\|.$$

The equality holds if and only, if one of u, v is a nonnegative multiple of the other.

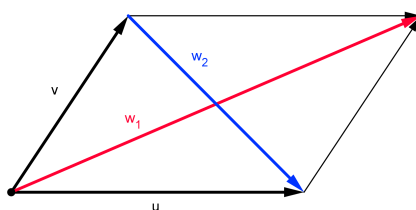


$$\vec{w} = \vec{u} + \vec{v}.$$

Parallelogram equality

If u, v are vectors in V , then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$



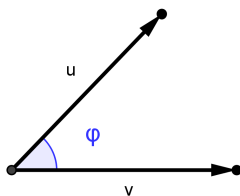
$$\vec{w}_1 = \vec{u} + \vec{v}, \quad \vec{w}_2 = \vec{u} - \vec{v}.$$

Definition

If u, v are vectors in V , $u \neq 0$ and $v \neq 0$, then we define the angle φ between vectors u and v as

$$\cos \varphi = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|},$$

where $\varphi \in \langle 0, \pi \rangle$.



Definition

A set of vectors is called *orthonormal* if the vectors are pairwise orthogonal and each vector has norm 1. An *orthonormal basis* of V is an orthonormal set of vectors in V that is also a basis of V .

Example

A standard orthonormal basis of the vector space \mathbb{R}^n is

$$B = \{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), (0, 0, 1, \dots, 0), \dots, (0, 0, 0, \dots, 1)\}$$

Gram-Schmidt theorem

If $\{v_1, v_2, \dots, v_m\}$ is a linearly independent set of vectors in V , then there exists an orthonormal set $\{e_1, \dots, e_m\}$ of vectors in V such that

$$\text{span}\{v_1, v_2, \dots, v_j\} = \text{span}\{e_1, \dots, e_j\}$$

for $j = 1, \dots, m$.

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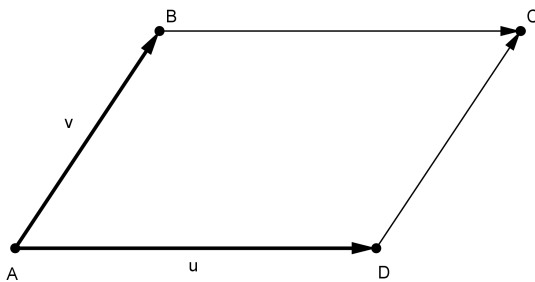
- **Vector product in \mathbb{R}^3**

Definition

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. By the *vector product* of u and v we mean the vector $w \in \mathbb{R}^3$ given by

$$u \times v = (u_1, u_2, u_3) \times (v_1, v_2, v_3) = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Note that the vector $u \times v$ is orthogonal to both vectors u and v .



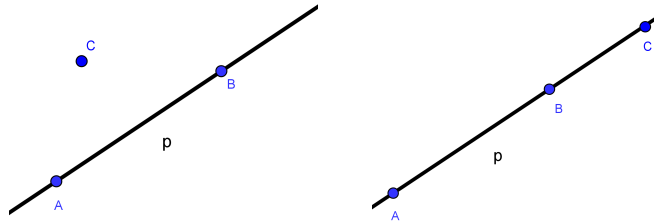
Note that the norm of the vector $u \times v$, that is $\|u \times v\|$, is equal to the area of parallelogram $ABCD$ given by vectors u and v .

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• **Some applications of determinants in analytic geometry**

Plane

1. Let $A = [x_1, y_1]$, $B = [x_2, y_2]$ and $C = [x_3, y_3]$ be three points in the plane V . Decide whether A , B and C lie on a single straight-line.



Calculate the determinant

$$\det A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

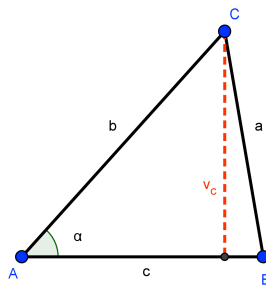
If $\det A = 0$, then the points A , B and C lie on a single straight-line. If $\det A \neq 0$, then the points A , B and C do not lie on any straight-line.

2. Let $A = [x_1, y_1]$ and $B = [x_2, y_2]$ be two points in the plane V . Write an analytic equation (so called general equation) of the straight-line AB .

Calculate the following determinant to obtain the analytic equation of the straight-line AB

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = (y_1 - y_2)x + (x_1 - x_2)y + x_1y_2 - x_2y_1 = 0.$$

3. Let $A = [x_1, y_1]$, $B = [x_2, y_2]$ and $C = [x_3, y_3]$ be three points in the plane V . Calculate the area of the triangle ABC .



Calculate the following determinant to obtain the area of the triangle ABC :

$$S = \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|.$$

Note

From the secondary school we know that the area of the triangle ABC can be calculated as

$$S = \frac{1}{2}c \cdot v_c = \frac{1}{2}b \cdot c \cdot \sin \alpha = \frac{1}{2} \|(\overrightarrow{AB}) \times \overrightarrow{AC}\|.$$

Space

1. Let $A = [x_1, y_1, z_1]$, $B = [x_2, y_2, z_2]$, $C = [x_3, y_3, z_3]$, $D = [x_4, y_4, z_4]$ be four points in \mathbb{R}^3 . Decide whether A , B , C and D lie on a single plane.

Calculate the determinant

$$\det A = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

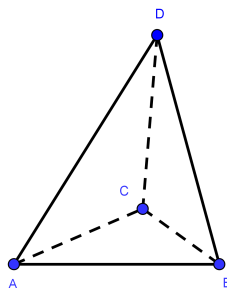
If $\det A = 0$, then the points A , B , C and D lie on a single plane. If $\det A \neq 0$, then the points A , B , C and D do not lie on any plane.

2. Let $A = [x_1, y_1, z_1]$, $B = [x_2, y_2, z_2]$ and $C = [x_3, y_3, z_3]$ be three points in \mathbb{R}^3 (not belonging to a single straight-line). Write the analytic equation (so called general equation) of the plane ABC .

Calculate the following determinant to obtain the analytic equation of the plane ABC

$$\det A = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x & y & z & 1 \end{vmatrix} = 0.$$

3. Let $A = [x_1, y_1, z_1]$, $B = [x_2, y_2, z_2]$, $C = [x_3, y_3, z_3]$ and $D = [x_4, y_4, z_4]$ be four points in \mathbb{R}^3 . Calculate the volume of the tetrahedron $ABCD$.



Calculate the following determinant to obtain the volume of the tetrahedron $ABCD$

$$V = \frac{1}{6} \left| \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \right|.$$

Note

From the secondary school we know that the volume of the tetrahedron $ABCD$ can be calculated as

$$V = \frac{1}{3} S_{ABC} \cdot v = \frac{1}{6} |(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}|,$$

where v is the distance of the point D from the plane ABC and S_{ABC} is the area of the triangle ABC .

Exercises

1. Calculate the following determinants

a)

$$\begin{vmatrix} 2 & 5 \\ -2 & 3 \end{vmatrix}.$$

b)

$$\begin{vmatrix} 1 & -3 \\ -4 & 6 \end{vmatrix}.$$

c)

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}.$$

d)

$$\begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix}.$$

e)

$$\begin{vmatrix} 1 & 0 \\ a & -2 \end{vmatrix}.$$

f)

$$\begin{vmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{vmatrix}.$$

g)

$$\begin{vmatrix} \sin x & -\sin y \\ \cos x & \cos y \end{vmatrix}.$$

h)

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & \sin x \end{vmatrix}.$$

i)

$$\begin{vmatrix} \tan x & -1 \\ 1 & \tan x \end{vmatrix}.$$

j)

$$\begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix}.$$

2. Calculate the following determinants

a)

$$\begin{vmatrix} 3 & -2 & 1 \\ -5 & 3 & 4 \\ 2 & 1 & 3 \end{vmatrix}.$$

b)

$$\begin{vmatrix} 4 & 10 & 1 \\ 0 & 2 & 0 \\ 1 & -3 & 7 \end{vmatrix}.$$

c)

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{vmatrix}.$$

d)

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}.$$

e)

$$\begin{vmatrix} 4 & 2 & 1 \\ 3 & -2 & -2 \\ 1 & 0 & 5 \end{vmatrix}.$$

f)

$$\begin{vmatrix} 5 & 0 & -1 \\ 2 & 4 & 0 \\ -3 & 6 & 1 \end{vmatrix}.$$

g)

$$\begin{vmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix}.$$

h)

$$\begin{vmatrix} -1 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{vmatrix}.$$

i)

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{vmatrix}.$$

j)

$$\begin{vmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{vmatrix}.$$

k)

$$\begin{vmatrix} 1 & 0 & f \\ u & 1 & k \\ 0 & 1 & k \end{vmatrix}.$$

l)

$$\begin{vmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{vmatrix}.$$

m)

$$\begin{vmatrix} a & a & a \\ -a & 0 & a \\ -a & -a & 0 \end{vmatrix}.$$

n)

$$\begin{vmatrix} a^2 + 1 & ab & ac \\ ab & b^2 + 1 & bc \\ ac & bc & c^2 + 1 \end{vmatrix}.$$

o)

$$\begin{vmatrix} \sin x & \cos x & 1 \\ \sin y & \cos y & 1 \\ \sin z & \cos z & 1 \end{vmatrix}.$$

3. Calculate the following determinants

a)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -5 & 2 & -1 & 1 \\ -6 & 5 & 2 & 1 \\ 3 & -1 & 1 & 0 \end{vmatrix}.$$

b)

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}.$$

c)

$$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}.$$

d)

$$\begin{vmatrix} -1 & -2 & 3 & -1 \\ 2 & 4 & -3 & 2 \\ 1 & 2 & -2 & -1 \\ -2 & 1 & 1 & -2 \end{vmatrix}.$$

e)

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

f)

$$\begin{vmatrix} 2 & 1 & 10 & 2 \\ 2 & 2 & -3 & 2 \\ -1 & 2 & 11 & 1 \\ 1 & 2 & 8 & 1 \end{vmatrix}.$$

g)

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 4 & 10 \\ 1 & 0 & 3 & -5 \\ 2 & 5 & 2 & 2 \end{vmatrix}.$$

h)

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & 2 & 2 & -2 \end{vmatrix}.$$

i)

$$\begin{vmatrix} -5 & 1 & -4 & 1 \\ 1 & 4 & -1 & 5 \\ -4 & 1 & -8 & -1 \\ 3 & 2 & 6 & 2 \end{vmatrix}.$$

j)

$$\begin{vmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix}.$$

k)

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix}.$$

l)

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix}.$$

m)

$$\begin{vmatrix} 0 & c & 1 & 0 \\ 1 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ 1 & 0 & -c & 1 \end{vmatrix}.$$

n)

$$\begin{vmatrix} 1 & 1 & 1 & a \\ 2 & 1 & 2 & b \\ 1 & -1 & 1 & c \\ 2 & 1 & -2 & d \end{vmatrix}.$$

4. Solve the equations

a)

$$\begin{vmatrix} x^2 & 3 & 2 \\ x & -1 & 1 \\ 0 & 1 & 4 \end{vmatrix} = 0.$$

b)

$$\begin{vmatrix} x^2 & 4 & 9 \\ x & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

c)

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & -b & b^2 \end{vmatrix} = 0.$$

d)

$$\begin{vmatrix} 1 & 3 & x \\ 3 & 1 & 5 \\ x & 2 & 10 \end{vmatrix} = 0.$$

5. Find the general equation of the straight-line AB and calculate the area of the triangle ABC

a) $A = [-1, 5], B = [2, -6], C = [4, 0].$

b) $A = [-1, 18], B = [1, 8], C = [2, 3].$

c) $A = [5, 0], B = [0, 2], C = [-2, -1].$

6. Find the general equation of the plane ABC and calculate the volume of the tetrahedron $ABCD$

a) $A = [3, 0, 4], B = [-1, -1, 7], C = [0, -2, -3], D = [6, 5, 4].$

b) $A = [3, 4, 5], B = [-2, -3, -4], C = [6, 0, 8], D = [3, 2, 7].$