II. MATRICES

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Definition

Let m and n be positive integers. By a *matrix of size m by n* (or an *$m \times n$ matrix*) we shall mean a rectangular array consisting of mn numbers displayed in m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}.$$

Note that the indexing is such that the first index gives the number of the *row* and the second index is that of the *column*, so that the entry a_{ij} appears at the intersection of the i-th row and the j-th column.

We shall often find it convenient to abbreviate the above array by

$$(a_{ij})_{m\times n} \quad \text{or} \quad A = (a_{ij})_{m\times n} \quad \text{or} \quad A = \big(a_{ij}\big), \quad i = 1, \dots, m,\ j = 1, \dots, n, \quad \text{or} \quad A.$$

Notes

1. When $m \neq n$, the matrix is called a *rectangular matrix.*
2. When $m = n$, the matrix is called a *square matrix.*
3. The numbers a_{ij} are from $\mathbb{R}$ or $\mathbb{C}$ for every $i = 1, \dots, m$, $j = 1, \dots, n$.
4. The elements a_{ij} for $i = j$ are called *diagonal elements of matrix.*

Examples

Let

$$A = \begin{pmatrix} -1 & 3 & -7 \\ 2 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix}, \qquad C = \begin{pmatrix} a & a & a \\ 0 & a & a \\ 0 & 0 & a \\ a & 0 & a \end{pmatrix}, \quad \text{where } a \in \mathbb{R}.$$

Matrix A is a matrix of size 2×3, matrix B is a matrix of size 3×3, matrix C is a matrix of size 4×3 matrix.

Before we develop an algebra for matrices, it is essential to know what is meant by saying that two matrices are *equal*. Common sense dictates that this should happen only if the matrices in question are of the same size and have the correspondig entries (numbers) equal.

Definition

If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$, then we say that A and B are *equal* (and wirte $A = B$) if,

(1) $m = p$ and $n = q$;

(2) $a_{ij} = b_{ij}$ for all i, j.

Note

The algebraic system that we shall develop for matrices will have many of the familiar properties enjoyed by the system of real numbers. However, as we shall see, there are also very striking differences.

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Elementary operations with matrices

Sum of matrices

Definition

Given $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we define the *sum* $A+B$ to be the $m \times n$ matrix whose (i, j)-th element is $a_{ij} + b_{ij}$.

Note that the sum $A + B$ is defined only when A and B are both of size $m \times n$; to obtain the sum we simply add the corresponding elements. Note also that $A + B$ is also an $m \times n$ matrix.

Examples

1. Let

$$A = \begin{pmatrix} -1 & 3 & 7 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 4 & 2 \end{pmatrix}.$$

Calculate $A + B$ and $A + C$.

- $A + B$ is defined because both matrices are of size 3×3.

$$A+B = \begin{pmatrix} -1 & 3 & 7 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix} = \begin{pmatrix} -1+1 & 3+1 & 7+1 \\ 2+2 & 0+4 & 2+8 \\ -1+3 & 2+9 & 0+27 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 8 \\ 4 & 4 & 10 \\ 2 & 11 & 27 \end{pmatrix}.$$

- $A + C$ is not defined because the matrix A is of size 3×3 and the matrix C is of size 2×3.

Properties of addition of matrices

Theorems

1. Addition of matrices is commutative [in the sense that if A, B are of the same size, then $A+B=B+A$].
2. Addition of matrice is associative [in the sense that if A, B, C are of the same size, then $A+(B+C)=(A+B)+C$].

The proofs are trivial. Using the definition of addition of matrices we obtain $a_{ij}+b_{ij}$ where $a_{ij}, b_{ij} \in \mathbb{R}$ (or $\mathbb{C}$) and ordinary addition of real or complex numbers is commutative and associative.

Theorem

There is a unique $m \times n$ matrix O such that $A+O=A$ for every $m \times n$ matrix A.

Example

For matrices of size 3×4 the matrix O is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Definition

The matrix O is called the $m \times n$ *zero matrix* and will be also denoted by $O_{m\times n}$. Thus $O_{m\times n}$ is the $m \times n$ matrix with all entries equal to 0.

Example

For matrix A of size 3×2

$$\begin{pmatrix} 1 & 0 \\ 2 & -2 \\ 11 & 3 \end{pmatrix}, \quad -A \quad \text{is the matrix} \quad \begin{pmatrix} -1 & 0 \\ -2 & 2 \\ -11 & -3 \end{pmatrix}.$$

Theorem

For every $m \times n$ matrix A there is a unique $m \times n$ matrix B such that $A+B=O$.

Definition

A unique matrix B satisfying $A+B=O$ is called the *additive inverse of* A and will be denoted by $-A$. Thus $-A$ is the matrix whose elements are the negatives of the corresponding elements of A.

Notes

• It is easily seen that for every $m \times n$ matrix A, we have $-(-A) = A$.

• So far, our matrix algebra has been confined to the operations of addition, which is a simple extension of the same notion for numbers.

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Matrix multiplication by scalars

Definition

Given a matrix A and a number λ, we define the *multiple of* A *by* λ to be the matrix, denoted by λA, that is obtained from A by multiplying every element of A by λ. Thus, if $A = (a_{ij})_{m\times n}$, then $\lambda A = (\lambda a_{ij})_{m\times n}$.

This operation is traditionally called the *multiple of a matrix by a scalar* (where the word *scalar* is taken to be synonymous with *number*).

Example

$$3 \cdot \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & -3 \\ 3 & -2 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 3 & 3 \\ 6 & 0 & -9 \\ 9 & -6 & 0 \end{pmatrix}.$$

Properties of matrix multiplication by scalars

The basic properties of this operation are listed in the following result.

Theorem

If A, B are $m \times n$ matrices, then, for all scalars λ and μ,

(1) $\lambda(A + B) = \lambda A + \lambda B$;

(2) $(\lambda + \mu)A = \lambda A + \mu A$;

(3) $\lambda(\mu)A = (\lambda\mu)A$;

(4) $(-1) \cdot A = -A$;

(5) $0 \cdot A = O_{m\times n}$.

Try to verify these statements using the definitions of operations.

Theorem

A set of matrices of size $m\times n$ with the „classical“ addition of matrices and the „classical“ multiplication by scalars is an example of a vector space.

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Matrix multiplication

We shall now describe the operation that is called *matrix multiplication*. This is the „multiplication“ of one matrix by another. At first glance this concept (due to A. Cayley) appers to be a most curious one. But it has in fact a very natural interpretation in an algebraic as well as geometrical contexts.

For the present, we shall simply accept it without asking how it arose. We will illustrate the later.

Definition

Let $A = (a_{ik})_{m\times n}$ and $B = (b_{kj})_{n\times p}$. Then we define the *product* AB to be the $m \times p$ matrix whose (i,j)-th element is

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Notes

To determine the (p,q)-th element of AB we multiply the corresponding elements of the p-th *row* of A by the corresponding elements in the q-th *column* of B and sum the products so formed.

It is important to note that there are no elements „left over“ in the sense that this sum of products is always defined, for in the definition of the matrix product AB the number n of columns of A is the same as the number of rows of B.

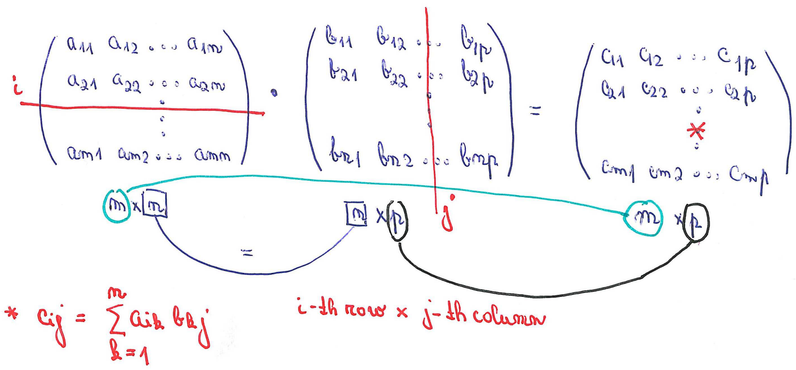

Illustration for matrix multiplication.

Examples

1. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 4 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Calculate AB and AC.

The product A and B is not definded because the matrix A is of size 2×2 and the matrix B is of size 3×2.

The product A and C is defined because the matrix A is of size 2×2 and the matrix C is of size 2×3. AC is the matrix of size 2×3.

$$AC = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1+0 & 2+2 & -1+6 \\ 0+0 & 0+1 & 0+3 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 5 \\ 0 & 1 & 3 \end{pmatrix}.$$

2. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Calculate AB and BA.

The product AB is defined since A is of size 2×3 and B is of size 3×2, hence AB is of size 2×2. We have

$$AB = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0+1+0 & 0+2+0 \\ 4+3+1 & 0+6+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 8 & 7 \end{pmatrix}.$$

Note that in this case the product BA is also defined (since B has the same number of columns as A has rows). The product BA is of size 3×3:

$$BA = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 0+0 & 2+0 & 0+0 \\ 0+4 & 1+6 & 0+2 \\ 0+2 & 1+3 & 0+1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 4 & 7 & 2 \\ 2 & 4 & 1 \end{pmatrix}.$$

The above example exhibits a „curious“ fact about matrix multiplication, namely that if AB and BA are defined then these products need not be equal. Indeed, as we have just seen, AB and BA need not be of the same size.

We now consider the basic properties of matrix multiplication.

Properties of matrix multiplication

Notes

Matrix multiplication is

(1) non-commutative [in the sense that, when the products are defined, $AB \neq BA$ in general];

(2) associative [in the sense that, when the products are defined, $A(BC) = (AB)C$].

Notes

Let A be a square matrix, $m \in \mathbb{N}$. By the symbol A^m we mean

$$\underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{\text{m-times } A}$$

• Matrix multiplication and matrix addition satisfy by the following *distributive laws.*

Theorem

When the relevant sums and products are defined, we hawe

$$A(B+C) = AB + AC \quad \text{and} \quad (B+C)A = BA + CA.$$

• Matrix multiplication and multiplication of matrix by scalars are related as follows.

Theorem

If AB is defined, then for all scalars λ we have

$$\lambda(AB) = (\lambda A)B = A(\lambda B).$$

• Other properties of matrix multiplication.

Theorem

There is a unique $n \times n$ matrix E such that $AE = EA = A$ for every $n \times n$ matrix A.

$$E = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

In other words, if we define the *Kronecker symbol* δ_{ij} by

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{otherwise,} \end{cases}$$

then $E = (\delta_{ij})_{n \times n}$.

Definition

The matrix E is called the $n \times n$ *identity matrix* and will be denoted by E.

Note

Note that E has all its diagonal entries equal to 1 and all other entries 0.

Definition

Let A, B be $n \times n$ matrices. Then A and B are said to *commute*, if $AB = BA$.

Examples

Let A be

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Find all matrices B which commute with matrix A.

From the definition we have $AB = BA$. It is obvious that the matrix B must be of size 2×2. So

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now we must calculate AB and BA

$$AB = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ a & b \end{pmatrix}.$$

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 2a \\ c+d & 2c \end{pmatrix}.$$

From the equality of AB and BA we obtain the following system of linear equations

$$\begin{aligned} a + 2c &= a + b, \\ b + 2d &= 2a, \\ a &= c + d, \\ b &= 2c. \end{aligned}$$

The solution of the above system of linear equations is $(c + d, 2c, c, d)$ where $c, d \in \mathbb{R}$. So that every matrix B which commutes with A is of the form

$$\begin{pmatrix} c+d & 2c \\ c & d \end{pmatrix}.$$

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Other information on matrices and their properties

Definition

A square matrix $D = (d_{ij})_{n \times n}$ is said to be *diagonal*, if $d_{ij} = 0$ whenever $i \neq j$. In other words, D is diagonal when all the entries off the main diagonal are 0.

Examples

The matrix of size 3×3

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

is diagonal.

Note

• It should be noted explicitely that there is no multiplicative analogue of the role of zero matrix, for example, if

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix},$$

so there is no matrix M such that $M \cdot A = E$.

Definition

If A is an $m \times n$ matrix, then by the *transpose* of A we shall mean the $n \times m$ matrix having (i, j)-th element equal to the (j, i)-th element of A. In other words, if $A = (a_{ij})_{m \times n}$, then the transpose of A is the $n \times m$ matrix, denoted by A^t such that $(a^t_{ij}) = a_{ji}$. (Note the interchange of indeces.)

Example

Find the transpose matrix of A and calculate AA^t and A^tA.

$$\text{If} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \end{pmatrix}, \qquad \text{then} \qquad A^t = \begin{pmatrix} 1 & 0 \\ 2 & -4 \\ 3 & 5 \end{pmatrix}$$

and

$$AA^t = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & -4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 7 \\ 7 & 41 \end{pmatrix},$$

$$A^tA = \begin{pmatrix} 1 & 0 \\ 2 & -4 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 20 & -14 \\ 3 & -14 & 34 \end{pmatrix}.$$

Basic properties of transposition are listed in the following result.

Theorem

When the relevant sums and products are defined, we have

1. $(A^t)^t = A$;
2. $(A + B)^t = A^t + B^t$;
3. $(\lambda A)^t = \lambda A^t$;
4. $(AB)^t = B^tA^t$ (note the exchange of order!).

Note

The first three equalities are immediate from the definition.

Definition

A square matrix A is said to be *symmetric*, if $A^t = A$; and *skew-symmetric* if $A^t = -A$.

Example

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & -3 \\ -1 & -3 & 0 \end{pmatrix} \text{ is a symmetric matrix, } \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} \text{ is a skew-symmetrix matrix.}$$

Definition

An $n \times n$ matrix is said to be *lower triangular*, if it is of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix},$$

thus if $a_{ij} = 0$ whenever $i < j$.

Definition

An $n \times n$ matrix is said to be *upper triangular*, if it is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

hence if $a_{ij} = 0$ whenever $i > j$.

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Elementary row operations on a matrix

(1) an interchange of two rows;
(2) a multiple of a row by a non-zero scalar;
(3) a sum of two rows.

Definition

By a *row-echelon* (or *stairstep*) matrix we mean a matrix of the form

$$\begin{pmatrix} 0 & \dots & 0 & * & & & & & \\ 0 & \dots & 0 & 0 & \dots & 0 & * & & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & * & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

in which every entry under the stairstep is 0, all of the entries marked $*$ are non-zero, and all other entries are arbitrary. (Note that the stairstep comes down one row at a time.) The entries marked $*$ will be called the *corner entries* of the stairstep.

Example

The 5×8 matrix

$$\begin{pmatrix} 0 & 1 & 3 & 0 & 2 & 9 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a row-echelon matrix.

Theorem

Every non-zero matrix A can be transformed into a row-echelon matrix by means of elementary row operations.

Example

$$\begin{pmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ -1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Definition

By a *linear combination* of the rows (columns) of A we shall mean an expression of the form

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p$$

where each x_i is a row (column) of A and every λ_i is a scalar.

Definition

If $x_1, x_2, \dots, x_p$ are rows (columns) of A, then we shall say that $x_1, x_2, \dots, x_p$ are *linearly independent* if

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p = o \Rightarrow \lambda_1 = \cdots = \lambda_p = 0.$$

In other words, the rows (columns) $x_1, x_2, \ldots, x_p$ are linearly independent if the only way how o can be expressed as a linear combination of $x_1, x_2, \ldots, x_p$ is the trivial one, namely

$$o = 0x_1 + 0x_2 + \cdots + 0x_p.$$

If $x_1, x_2, \ldots, x_p$ are not linearly independent, then we say they are *linearly dependent.*

Theorems

- If the rows (columns) $x_1, x_2, \ldots, x_p$ are linearly independent, then none of them can be zero.
- The rows (columns) $x_1, x_2, \ldots, x_p$ are linearly dependent, if and only if at least one can be expressed as a linear combination of the others.
- The rows of a matrix are linearly dependent, if and only if one can be obtained from the others by means of elementary row operations.

Definition

By the *row rank* of a matrix we mean the maximum number of linearly independent rows in the matrix.

Definition

A matrix B is said to be *row-equivalent* to the matrix A, if B can be obtained from A by a finite sequence of elementary row operations.

Theorems

(1) Elementary row operations do not affect row rank.

(2) If B is any row-echelon form of A, then B has the same row rank as A.

(3) The row rank of a matrix is the number of non-zero rows in any row-echelon form of the matrix.

(4) Row-equivalent matrices have the same rank.

(5) Both row and column rank are invariant with respect to both row and column elementary operations.

(6) Row rank and column rank are equal.

Note

An *elementary column operation* is defined analogously. Simply replace „row“ by „column“ in the definition.

Example

Let

$$A = \begin{pmatrix} 1 & 2 & -3 & 3 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 3 & 3 \end{pmatrix}.$$

Calculate the row rank of the matrix A.

We perform a sequence of elementary row operations to obtain a row-echelon matrix. From a row-echelon matrix we will obtain the maximum number of linearly independent rows in the matrix.

$$\begin{pmatrix} 1 & 2 & -3 & 3 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & -4 & 7 & -6 \\ 0 & -2 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 3 \\ 0 & -4 & 7 & -6 \\ 0 & 0 & -5 & -6 \end{pmatrix}.$$

As we see, in the row-echelon matrix, there are 3 linearly independent rows, so the matrix A has the row rank 3.

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Invertible matrices

Definition

Let A be an $n \times n$ matrix. An $n \times n$ matrix X is said to be the *inverse* of A if $AX = XA = E$. We will denote it by the symbol A^{-1}.

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

(1) A has a inverse;
(2) A is of rank n.

Note

A matrix A has at most one inverse matrix.

Theorem

(1) We have $(A^{-1})^{-1} = A$ for every invertible matrix A.
(2) Let A, B be $n \times n$ matrices. If A and B are invertible, then so is AB; moreover, $(AB)^{-1} = B^{-1}A^{-1}$.
(3) If A is invertible, then so is A^m for every positive integer m; moreover, $(A^m)^{-1} = (A^{-1})^m$.
(4) If A is invertible, then so is its transpose; we have $(A^t)^{-1} = (A^{-1})^t$.

An algorithm for finding an inverse matrix

$(A|E)$

$\rightarrow$ make elementary row operations to obtain an *upper triangle matrix* $\rightarrow$

$\rightarrow$ make elementary row operations to obtain a *diagonal matrix* $\rightarrow$

$(E|A^{-1})$.

Examples

1. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

.

Find A^{-1}.

The inverse matrix is not defined, since A is not a square matrix.

2. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 8 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

.

Find A^{-1}.

The algorithm yields

$$\begin{pmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 4 & 8 & 0 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & -4 & 1 & 0 \\ 0 & -1 & 1 & | & -1 & 0 & 1 \end{pmatrix}.$$

We see that A is a matrix of size 3×3, but its row rank is only 2. The inverse matrix does not exist.

3. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

.

Find A^{-1}.

By elementary row operations we get

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 1 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -1 & -2 & | & -1 & 1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -1 & -2 & | & -1 & 1 & 0 \\ 0 & 0 & -2 & | & -1 & 1 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 2 & 4 & 0 & | & -1 & 3 & 3 \\ 0 & -1 & 0 & | & 0 & 0 & -1 \\ 0 & 0 & -2 & | & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & | & -1 & 3 & -1 \\ 0 & -1 & 0 & | & 0 & 0 & -1 \\ 0 & 0 & -2 & | & -1 & 1 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & | & 0 & 0 & 1 \\ 0 & 0 & 1 & | & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 3 & -1 \\ 0 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix}.$$

To verify the result, we show that

$$A \cdot A^{-1} = A^{-1} \cdot A = E \quad \text{or} \quad (A^{-1})^{-1} = A.$$

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Exercises

1. Calculate the products AB and BA (if exist):

a)

$$A = \begin{pmatrix} 2 & 5 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ -4 & 6 \end{pmatrix}.$$

b)

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 2 & -1 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 \\ -5 & 1 \\ 2 & -3 \\ 5 & 2 \end{pmatrix}.$$

c)

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 2 & 0 & 5 \\ 3 & -1 & -4 \\ 1 & 6 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 2 & -4 & 3 & 1 \\ 0 & 5 & -2 & 1 \end{pmatrix}.$$

d)

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 3 & -2 & -2 \\ 1 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 0 & -1 \\ 2 & 4 & 0 \\ -3 & 6 & 1 \end{pmatrix}.$$

2. Calculate $AB - BA$ (if exists):

a)

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{pmatrix}.$$

b)

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}.$$

3. Calculate

a)

$$\begin{pmatrix} 4 & 2 & 1 \\ 3 & -2 & -2 \\ 1 & 0 & 5 \end{pmatrix}^2 .$$

b)

$$\begin{pmatrix} 5 & 0 & -1 \\ 2 & 4 & 0 \\ -3 & 6 & 1 \end{pmatrix}^2 .$$

c)

$$\begin{pmatrix} 2 & 5 & -1 \\ 1 & 0 & -4 \\ -3 & 1 & 2 \end{pmatrix}^2 .$$

d)

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 1 & -3 \end{pmatrix}^4 .$$

e)

$$\begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix}^3 .$$

f)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{129} .$$

4. Calculate the products AA^t and A^tA:

a)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ -3 & 1 & 1 \end{pmatrix} .$$

b)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 1 \end{pmatrix} .$$

c)

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -2 & 1 & 0 \end{pmatrix} .$$

d)

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 0 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix} .$$

e)

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}.$$

5. Calculate the row rank of matrices:

a)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

b)

$$\begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix}.$$

c)

$$\begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

d)

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

e)

$$\begin{pmatrix} 1 & 3 & 4 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 2 & 4 & 4 \end{pmatrix}.$$

f)

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 6 & 1 & 2 & 3 \\ 3 & 2 & 4 & 5 \\ 4 & 6 & 1 & 3 \end{pmatrix}.$$

g)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 1 \\ 1 & 3 & 6 & 10 & 1 \\ 1 & 4 & 10 & 20 & 1 \end{pmatrix}.$$

h)

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 1 & 3 \\ 2 & 6 & 1 & 10 & 0 & 1 \\ -1 & -3 & 0 & -4 & 1 & 0 \\ 5 & 15 & -9 & 2 & 3 & 12 \end{pmatrix}.$$

i)

$$\begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & -1 & 3 \\ -1 & 1 & 2 & -2 \\ 3 & 4 & -2 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

j)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & -1 & 5 \\ 1 & 3 & 4 & 5 & 5 & 8 \\ 2 & 5 & 9 & 8 & 7 & 12 \\ 2 & 6 & 8 & 10 & 13 & 18 \\ 3 & 8 & 15 & 12 & 15 & 21 \end{pmatrix}.$$

6. Calculate the matrix A^{-1} (if exists):

a)

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

b)

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

c)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}.$$

d)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

e)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}.$$

f)

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix}.$$

g)

$$A = \begin{pmatrix} -2 & 2 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix}.$$

h)

$$A = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

i)

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 7 & 0 & 1 & 1 \\ -17 & 0 & -1 & 0 \end{pmatrix}.$$

j)

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

7. Solve the matrix equation for a matrix X:

a)

$$X \cdot \begin{pmatrix} 3 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 2 & 7 \\ 8 & 6 & -5 \end{pmatrix}.$$

b)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 5 \\ 7 & 1 \end{pmatrix}.$$

c)

$$\begin{pmatrix} 1 & 2 & 4 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -5 & 2 & -1 & 1 \\ -6 & 5 & 2 & 1 \\ 3 & -1 & 1 & 0 \end{pmatrix}.$$

d)

$$\begin{pmatrix} 2 & 4 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 4 \\ -5 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

e)

$$X \cdot \begin{pmatrix} 0 & 4 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 4 \\ -5 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 3 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

f)

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 5 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & 0 \\ -1 & 2 & -3 \end{pmatrix}.$$