Penalization

$$[y_t, u_t, y_{t-1}, u_{t-1}, 1] \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} & \Omega_{35} \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & \Omega_{45} \\ \Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} \end{bmatrix} \begin{bmatrix} y_t \\ u_t \\ y_{t-1} \\ u_{t-1} \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \Omega_{11} y_t, & \Omega_{12} y_t, & \Omega_{13} y_t, & \Omega_{14} y_t, & \Omega_{15} y_t \\ + & + & + & + & + \\ \Omega_{21} u_t, & \Omega_{22} u_t, & \Omega_{23} u_t, & \Omega_{24} u_t, & \Omega_{25} u_t \\ + & + & + & + & + \\ \Omega_{31} y_{t-1}, & \Omega_{32} y_{t-1}, & \Omega_{33} y_{t-1}, & \Omega_{34} y_{t-1}, & \Omega_{35} y_{t-1} \\ + & + & + & + & + \\ \Omega_{41} u_{t-1}, & \Omega_{42} u_{t-1}, & \Omega_{43} u_{t-1}, & \Omega_{44} u_{t-1}, & \Omega_{45} u_{t-1} \\ + & + & + & + & + \\ \Omega_{51}, & \Omega_{52}, & \Omega_{53}, & \Omega_{54}, & \Omega_{55} \end{bmatrix} \begin{bmatrix} y_t \\ \\ u_t \\ \\ y_{t-1} \\ \\ u_{t-1} \\ \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \Omega_{11} y_t^2 + & \Omega_{12} y_t u_t + & \Omega_{13} y_t y_{t-} + & \Omega_{14} y_t u_{t-1} + & \Omega_{15} y_t \\ + & + & + & + & + \\ \Omega_{21} u_t y_t + & \Omega_{22} u_t^2 + & \Omega_{23} u_t y_{t-1} + & \Omega_{24} u_t u_{t-1} + & \Omega_{25} u_t \\ + & + & + & + & + \\ \Omega_{31} y_{t-1} y_t + & \Omega_{32} y_{t-1} u_t + & \Omega_{33} y_{t-1}^2 + & \Omega_{34} y_{t-1} u_{t-1} + & \Omega_{35} y_{t-1} \\ + & + & + & + & + \\ \Omega_{41} u_{t-1} y_t + & \Omega_{42} u_{t-1} u_t + & \Omega_{43} u_{t-1} y_{t-1} + & \Omega_{44} u_{t-1}^2 + & \Omega_{45} u_{t-1} \\ + & + & + & + & + \\ \Omega_{51} y_t + & \Omega_{52} u_t + & \Omega_{53} y_{t-1} + & \Omega_{54} u_{t-1} + & \Omega_{55} \end{bmatrix}$$

$$J = (y_t - s_t)^2 + \omega u_t^2 + \lambda (u_t - u_{t-1})^2 =$$

$$= y_t^2 - 2 y_t s_t + s_t^2 + \omega u_t^2 + \lambda u_t^2 - 2\lambda u_t u_{t-1} + \lambda u_{t-1}^2$$

$$y_t, \;\; u_t, \;\; y_{t-1}, \;\; u_{t-1}, \;\; 1$$

$$\begin{matrix} y_t \\ u_t \\ y_{t-1} \\ u_{t-1} \\ 1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -s_t \\ 0 & \omega + \lambda & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & \lambda & 0 \\ -s_t & 0 & 0 & 0 & s_t^2 \end{bmatrix}$$

Completion to squares

Scalar case

$$(u+m)^2 = u^2 + 2um + m^2$$

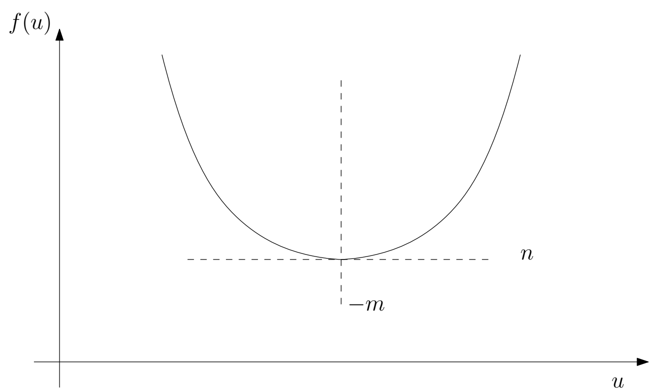
this is a template

Example

$$au^2 + 2bu + c = a\left(u^2 + 2u\frac{b}{a}\right) + c =$$

$$= a\left(u^2 + 2u\underbrace{\frac{b}{a}}_{m} + \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2\right) + c =$$

$$a\left(u + \underbrace{\frac{b}{a}}_{m}\right)^2 + \underbrace{c - \frac{b^2}{a}}_{n}$$



Vector case

analogy $-\ au^2 \to u'Au, \qquad (Au)' = u'A'$

E$u$ample
A square symmetric

$$u'Au + 2u'Bx + x'Cx =$$

$$= u'Au + 2u'A\underbrace{A^{-1}Bx}_{m} + \underbrace{\left(A^{-1}Bx\right)' A \left(A^{-1}Bx\right)}_{m'Am} - \underbrace{\left(A^{-1}Bx\right)' A \left(A^{-1}Bx\right)}_{m'Am} + x'Cx =$$

$$= \left(u + A^{-1}Bx\right)^{'} A \left(u + A^{-1}Bx\right) + x'B'A^{-1}AA^{-1}Bx + x'Cx =$$

$$\left(u + A^{-1}Bx\right)^{'} A \left(u + A^{-1}Bx\right) + x' \left(B'A^{-1}Bx + C\right) x$$

Expectation

$$E\left[x_t^{'}x_t|u_t, d\left(t-1\right)\right] = E\left[\left(Mx_{t-1} + Nu_t + w_t\right)' \left(Mx_{t-1} + Nu_t + w_t\right)\right] =$$

$$= \left(Mx_{t-1} + Nu_t\right)' \left(Mx_{t-1} + Nu_t\right) + 2\left(Mx_{t-1} + Nu_t\right)' E\left[w_t\right] + E\left[w_t^{'}w_t\right]$$

$$= \left(Mx_{t-1} + Nu_t\right)' \left(Mx_{t-1} + Nu_t\right) + E\left[w_t^{'}w_t\right] =$$

$$\left(x_{t-1}^{'}M' + u_t^{'}N'\right)\left(Mx_{t-1} + Nu_t\right) + r =$$

$$= x_{t-1}^{'}M'Mx_{t-1} + 2x_{t-1}^{'}M'Nu_t + u_t^{'}N'Nu_t + r$$

Completion to square

$$\text{formula} \quad (x+a)^2 = x^2 + 2xa + a^2$$

$$x^2 + xm + c = x^2 + 2x\frac{m}{2} + \left(\frac{m}{2}\right)^2 - \left(\frac{m}{2}\right)^2 + c =$$

$$= \underbrace{\left(x + \frac{m}{2}\right)^2}_{\text{square}} + \underbrace{c - \left(\frac{m}{2}\right)^2}_{\text{reminder}}$$